Optimal Contracts with Randomly Arriving Tasks*

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Abstract

Workers are rarely assigned to perform the same task throughout their career. Instead, their assignments may change randomly over time to comply with the fluctuating needs of the organisation where they are employed. In this paper, we show that this typical randomness in workplaces has a striking effect on the structure of long-term employment contracts. In particular, simple intertemporal variability in the worker's tasks is sufficient to generate a rich promotion-based dynamics in which, occasionally, the worker receives a (permanent) wage raise and his future work requirements are reduced.

Keywords: Dynamic contracting, random tasks, seniority, promotion.

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1 Introduction

Promotions and seniority-based dynamics are widespread in workplaces. Modifications of work requirements and responsibilities, as well as wage raises, are often seen throughout a worker's tenure. In this paper, we link these observations to a feature that is common to many workplaces but has received scant attention in the literature: the tasks to which a worker is assigned change over time due to the varying needs of the organisation where he is employed. We show that simple variability in the worker's tasks generates a rich promotion-based dynamics. Most of the time, the worker's wage is constant despite the fluctuations in his assignments. However, occasionally, the worker receives a permanent wage raise, and his future work requirements are reduced.

We analyse a simple and transparent model where different production opportunities arrive randomly in a manner that is i.i.d. across periods. Each period, the agent can exert effort whose productivity depends on the realised production opportunity. The principal incentivizes the agent via a periodic wage. We assume that the marginal productivity of effort, as well as the worker's marginal utility from the wage, are decreasing. Finally, we abstract away from frictions that arise from informational asymmetries and assume that both the worker's effort and the arrival of production opportunities are perfectly observed.

We show that, even though production opportunities arrive according to a stationary distribution, the unique optimal employment contract consists of multiple hierarchical phases, which we interpret as different "ranks." Within each rank, the worker receives a constant periodic wage. His workload, however, is stochastic since the effort required depends on the opportunities that are available in each period. Promotion to a new (higher) rank occurs upon the arrival of a production opportunity for which the worker's effort is more productive relative to all previous opportunities. On that occasion, the worker enjoys not only a permanent wage raise but also a reduction in his future workload. That is, whenever opportunities similar to those that have been available in the past arise again, the worker will be instructed to work less.

In general, as time goes by, the worker's expected effort within a period decreases while his periodic wage increases. Our analysis, therefore, offers new insights into wage ladders

and seniority in workplaces by drawing a clear connection between the intertemporal variability of on-the-job assignments and the dynamics of effort and compensation. Specifically, sophisticated promotion dynamics may arise efficiently even in the absence of traditional frictions such as imperfect or asymmetric information about the worker's ability or actions, search frictions in the labour market, bargaining, accumulation of expertise, etc.

In addition to the aforementioned properties of optimal employment contracts, our results also provide a rationale for wage stickiness, and can thus be linked to the macroeconomics literature that studies the volatility of employment and productivity. Shimer (2005) showed that a reasonably calibrated textbook search-and-matching model (Diamond, 1982; Mortensen and Pissarides, 1994) cannot explain the volatility of the vacancy-unemployment ratio in the US data. In response, Hall (2005) argued that this puzzle can be resolved if wages are sticky. To support this line of argument, one might consider the varying opportunities in our model as "productivity shocks" of an aggregate nature. Since wages do not respond to negative productivity shocks under the optimal contract, we obtain a complete *downward* wage rigidity. Also, since wages respond to positive productivity shocks only when the productivity shock is unprecedentedly high, we obtain a partial *upward* wage rigidity. Thus, our model can provide a parsimonious theoretical foundation (that relies on nothing but the productivity shocks themselves and the firm's ability to offer long-term contracts) for the wage stickiness required to sustain Hall's argument.[1]

1.1 Related Literature

Many explanations have been proposed for the phenomenon of seniority in workplaces, and especially the increasing wage dynamics. For example, Becker (1962) and Parsons (1972) emphasise the effect of relationship-specific investment, Freeman and Medoff (1984) highlight the role of labour unions and collective bargaining, and Lazear (1981) and Carmichael (1983) illustrate how increasing wage dynamics emerge from informational frictions. More closely related to seniority-based dynamics are the papers that derive a downward wage rigidity in stochastic employment environments with symmetric information: Harris and Holmström (1982) study labour markets where there is uncertainty about the worker's skill,

[1] Pissarides (2009) argues that wage stickiness can explain the "Shimer puzzle" only if the wage of a newly hired worker is sticky. Generically, aggregate productivity is not at its lowest when a worker is hired, and so our model predicts that the wage of a newly hired worker is sticky.

Holmström (1983) considers markets with stochastic demand, and Postel-Vinay and Robin (2002a,b) analyse labour markets with search frictions.[2]

Our work differs from the strand of the literature discussed above in three key aspects. First, in addition to the dynamic structure of wage, our paper is informative about the dynamics of the worker's effort and the allocation of effort between various types of tasks along the worker's career. Second, in the previously mentioned papers, the downward wage rigidity is a result of competition for the worker's service. By contrast, the wage and effort dynamics in the present paper are entirely an outcome of within-interaction efficiency considerations (the worker's outside option is constant). Finally, in the previously mentioned papers, modifications in contract terms are triggered by purely exogenous events, whereas in our work promotions are driven endogenously.

The decreasing effort trajectory has been addressed recently in the literature. For example, Fudenberg and Rayo (2019) document that apprentices provide unskilled labour to their employers as "payment" for their training, and explain why these demands decrease over time. In addition, Barlevy and Neal (2019) argue that recently hired associates in law firms work long hours in order to generate information about whether or not they are suitable for promotion to a partnership position. Moreover, they document that their working hours decrease once they are no longer eligible for that promotion. We contribute to the discussion on the workers' workload dynamics by offering an alternative rationale for its reallocation and decrease over time that relies on nothing but fluctuations in the contracting environment.

In a different strand of literature, Marcet and Marimon (1992) and Krueger and Uhlig (2006) study insurance markets with one-sided commitment. They show that changes in the terms of insurance contracts always favour the insured due to a mechanism similar to the one in Harris and Holmström (1982). A more recent strand of literature studies dynamic project selection in environments where projects arrive randomly (e.g., Li *et al.*, 2017; Samuelson and Stacchetti, 2017; Bird and Frug, 2019; Forand and Zápal, 2020; Lipnowski and Ramos, 2020). In those papers, projects are represented by pairs of payoffs and the players decide

[2]Thomas and Worrall (1988) study wage dynamics when firms cannot sign binding contracts and the value of a worker's output varies over time. They find that, generally, wage dynamics is nonmonotone.

whether or not to implement an available project.[3] With the exception of Forand and Zápal (2020), these papers assume that there is asymmetric information and/or that the principal lacks commitment power, and find that project selection criteria can be nonmonotone.

Forand and Zápal (2020) consider a model where the principal has full commitment power, projects arrive according to a general stochastic process, and the available project is publicly observed. They establish the following result. Any project that generates a positive (negative) payoff for both players should be implemented (discarded); projects for which the agent's payoff is negative and the principal's payoff is positive should be implemented (discarded) if the absolute value of the ratio of these payoffs is below (above) a threshold that *does not increase over time*; finally, projects that generate a positive payoff for the agent and a negative payoff for the principal should be implemented (discarded) when the absolute value of the ratio of these payoffs is above a threshold that *does not increase over time*. Similar to the present paper, their project selection criteria evolve monotonically in the agent's favour over time. Unlike our paper, due to the generality of their project arrival process, they do not solve for the optimal contract and their characterisation is silent about the nature of the events that may trigger changes in the above-mentioned thresholds. In addition, unlike our paper, due to the linearity of payoffs, there is a bang-bang solution for all "off-threshold" projects, and the dynamics of threshold projects cannot be uniquely determined.

Finally, Ray (2002), Thomas and Worrall (2018), and Bird and Frug (2020) consider abstract contracting environments and derive general monotonicity results. In particular, Ray (2002) studies contracting with limited commitment in a repeated (deterministic) interaction, Thomas and Worrall (2018) consider relational contracting in stochastic hold-up problems, and Bird and Frug (2020) study contracting with full commitment in general stochastic environments. Due to the generality of these papers, only a partial characterisation of optimal contracts is derived that is not informative about important qualitative aspects of the contract. By contrast, the present paper imposes a structure that enables us to solve for the unique optimal contract and show how the random arrival of production opportunities leads to a promotion-based dynamics.

[3] Armstrong and Vickers (2010) study project selection criteria in a single-period model.

2 Model

We consider an infinitely repeated interaction between an agent and a principal. At the beginning of each period, nature chooses a task from the set $\mathcal{I} = \{1, 2, ..., I\}$. Task $i \in \mathcal{I}$ is drawn with probability $q_i > 0$, $\sum_{i=1}^{I} q_i = 1$. The realised task (also referred to as the available task) is observed by both players. After observing which task is available, the agent chooses the effort $e \in [0, \infty)$ and, after observing this choice, the principal pays wage $w \in [0, \infty)$. The events in period t are fully summarised by the triplet (i_t, e_t, w_t).

We denote the principal's profit from effort on task $i \in \mathcal{I}$ by $\pi_i(\cdot)$, and the agent's utility from wage by $g(\cdot)$. We assume that the agent's and principal's periodic payoffs given the events (i, e, w) are, respectively, $g(w) - e$ and $\pi_i(e) - w$. Moreover, we assume that the functions $g(\cdot), \pi_1(\cdot), \ldots, \pi_I(\cdot)$ are continuously differentiable, strictly concave, increasing, and satisfy $g(0) = \pi_i(0) = 0$. In addition, we assume that tasks are ordered with respect to the marginal productivity of effort: for all $i < I$ and $e \in [0, \infty)$, $\pi_i'(e) < \pi_{i+1}'(e)$. Finally, to avoid trivialities, we restrict attention to situations where all tasks are potentially profitable and it is suboptimal to incentivize infinite effort. To do so (due to the ordering assumption), it is sufficient to impose $\pi_1'(0) > \frac{1}{g'(0)}$ and $\lim_{e \to \infty} \pi_I'(e) < \lim_{w \to \infty} \dfrac{1}{g'(w)}$.

A sequence of triplets $h_t = \{(i_s, e_s, w_s)\}_{s<t}$, where, for all $s < t$, $i_s \in \mathcal{I}$ and e_s, w_s are nonnegative numbers, represents a generic history at the beginning of period t. We denote by H_t the set of all possible histories at the beginning of period t, by $H = \cup_{t \in \mathbb{N}} H_t$ the set of all "beginning-of-period" histories, by $(h_t; i_t)$ a generic history at which the agent chooses the effort of period t, and by $(h_t; i_t, e_t)$ a generic history at which the principal chooses the wage of period t. The players use the same factor $\delta < 1$ to discount future payoffs, and their objective is to maximise the sum of their discounted payoffs.

At the beginning of the interaction (prior to the realisation of i_1) the principal proposes a contract, to which he commits. A contract $\langle work, pay \rangle$ consists of two functions: a job

description

$$work : H \times \mathcal{I} \to [0, \infty)$$

that specifies the required effort in period t as a function of the history at the beginning of the period and the available task; and a compensation plan

$$pay : H \times \mathcal{I} \times [0, \infty) \to [0, \infty)$$

that specifies the agent's wage in period t as a function of the history at the beginning of the period, the available task, and the agent's choice of effort.

The agent does not have commitment power. Thus, he will follow the contract only if doing so maximises his expected discounted utility at every (h_t, i_t). A contract $\langle work, pay \rangle$ under which the agent finds it optimal to choose effort in accordance with $work(\cdot)$ is called *incentive compatible*. For the rest of the paper we restrict attention to incentive-compatible contracts. Since the agent can always choose to exert no effort and his utility from wage is nonnegative, he can guarantee himself a continuation payoff of zero after any possible history. Therefore, an explicit individual rationality constraint is unneeded.

It is worth pointing out that if we replace the concavity assumptions on $g(\cdot)$ and $\pi_i(\cdot)$ with their weak versions, the contract we construct in Section 3.2 remains an optimal contract, albeit not necessarily the unique optimal contract. For example, in the extreme case of linear utility from wage, a trivial optimal contract exists where the principal fully compensates the agent at the end of each period. While that contract seems natural in the case of linear utility, it cannot be approximated as a limit of optimal contracts where the agent's utility from wage is strictly concave. It is the strict concavity of the agent's utility that constitutes the link between different periods in our model.

3 Main Result

In this section we characterise the unique optimal contract. Figure 1 visualises the dynamics of wage and effort under this contract for a typical sequence of task arrivals. The upper panel depicts the arrival of tasks along a specific history (exogenous data, not determined by

the contract). The middle and bottom panels depict, respectively, the endogenous response of wage and effort to the chaotic arrival of tasks along the same history.

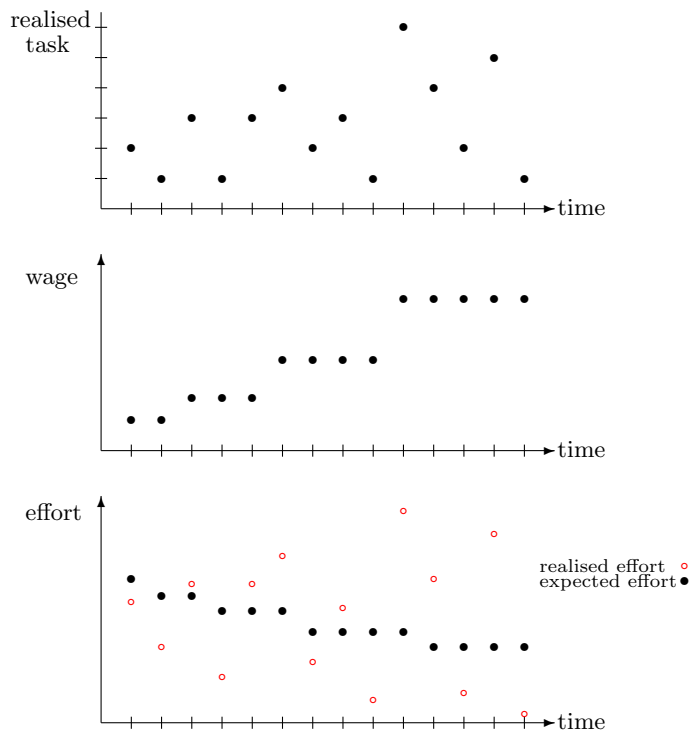

Figure 1: Qualitative dynamics of wage and effort for the task arrival sequence $(2, 1, 3, 1, 3, 4, 2, 3, 1, 6, 4, 2, 5, 1)$.

The agent's actual workload in every period (red circles) suggests that a great deal of the exogenous uncertainty carries over to the optimal contract. However, the agent's expected workload before each period's task is realised and his actual wage (both represented by black dots) show that, in fact, the contract generates well-behaved monotonic patterns. A closer look at Figure 1 reveals that the agent's wage increases upon the arrival of a task that is better than all previously available tasks, and that his expected workload decreases in the subsequent period.[4] Our main result (which follows formally from Proposition 2 in Section 3.2) will establish that this is a general feature of the unique optimal contract. For every $h_t \in H$ and $i_t \in \mathcal{I}$, let $\mathscr{I}(h_t; i_t) = max\{i_s : s \leq t\}$ denote the type of the best-to-date task.

[4]We say that a task of type i is better than a task of type j if $i > j$.

Proposition 1. *The agent's wage is an increasing function that depends only on $\mathscr{I}(\cdot;\cdot)$, and his effort on any type of task is a decreasing function that depends only on $\mathscr{I}(\cdot;\cdot)$.*

As the function $\mathscr{I}(\cdot;\cdot)$ does not decrease over time, Proposition 1 has the following corollary.

Corollary 1. *The worker's career path exhibits advancement through a monotonically increasing sequence of ranks, his periodic wage is constant within each rank and increases upon each promotion, and his effort on every type of task decreases over time.*

3.1 Auxiliary Problems

The key step in characterising the optimal contract is to show that whenever the available task is better than all previously available tasks, no debt should carry over from the past (i.e., the agent's continuation utility should be zero). Given this insight (which we establish below), the whole interaction can be split into parts that can be analysed separately. We now define I auxiliary problems, one for each type of task, that constitute the building blocks of the optimal contract.

For $i \in \mathcal{I}$, let $P^{(i)}$ (referred to as "auxiliary problem i") denote the principal's optimisation problem in an auxiliary setting where:

(1) At $t = 1$ the realised task is of type i. In all other periods, task $j \in \mathcal{I}$ is realised with probability q_j.

(2) The interaction terminates upon the first arrival of a task that is better than i (note that neither effort nor compensation can be provided after the arrival of such a task).

(3) The principal must hire the agent on fixed terms. A stationary contract $\langle (e_j)_{j \leq i}, w \rangle$ specifies the required effort e_j whenever a task of type j is available, for all $j \in \{1, .., i\}$, and the constant wage w that the agent receives in each period so long as he has followed the contract in the past. If the agent has not followed the contract in the past, his wage is zero.

For a contract to be incentive compatible in this auxiliary setting, the agent's cost of effort on the available task must not exceed his payoff from the current period's wage plus

the expected discounted payoff from wage and effort in future periods. Let

$$\lambda_i = \sum_{j \leq i} q_j$$

denote the probability that the realised task is not better than i. Hence, the probability that the interaction in auxiliary problem $P^{(i)}$ is still ongoing in period $t+1$ is λ_i^t. The incentive-compatibility constraint when a task of type j is available in auxiliary problem $P^{(i)}$ is

$$e_j \leq g(w) + \sum_{s=1}^{\infty} (\lambda_i \delta)^s \left(g(w) - \frac{1}{\lambda_i} \sum_{k \leq i} q_k e_k \right),$$

which simplifies to

$$e_j \leq \frac{1}{1 - \delta \lambda_i} \left(g(w) - \delta \sum_{k \leq i} q_k e_k \right). \qquad (IC_j^{(i)})$$

Thus, for $i \in \mathcal{I}$, auxiliary problem $P^{(i)}$ is given by

$$P^{(i)} = \left\{ \begin{array}{c} \max_{w,(e_j)_{j \leq i}} \left[\pi_i(e_i) + \frac{1}{1-\delta\lambda_i} \left(\delta \sum_{j \leq i} q_j \pi_j(e_j) - w \right) \right] \\ s.t. \quad IC_1^{(i)}, \ \ldots, IC_i^{(i)} \end{array} \right\}.$$

Since $P^{(i)}$ is a convex optimisation problem, it has a unique solution, which we denote by $\langle (e_j^{(i)})_{j \leq i}, w^{(i)} \rangle$. We now derive several important properties of the solutions to the auxiliary problems; we will later use these properties to establish the optimality of the contract we construct.

Lemma 1. *The only binding constraint in the solution to $P^{(i)}$ is $IC_i^{(i)}$.*

Proof. In optimum, $w^{(i)} > 0$ since $\pi_i'(0) > \frac{1}{g'(0)}$. Moreover, at least one of the incentive-compatibility constraints must be binding since, otherwise, slightly reducing $w^{(i)}$ would increase the value of the problem without violating any of the constraints.

Assume by way of contradiction that $IC_j^{(i)}$ is binding, for some $j < i$. This implies that $e_j^{(i)} > 0$ and, since the right-hand side of all the constraints in $P^{(i)}$ are identical, it follows that $e_j^{(i)} \geq e_i^{(i)}$.

Consider the following modification for $\epsilon > 0$: decrease $e_j^{(i)}$ by $\epsilon \frac{1-\delta\lambda_i}{\delta q_j}$ and increase $e_i^{(i)}$ by $\epsilon(1 + \frac{\delta q_i}{1-\delta\lambda_i})^{-1}$. It is straightforward to verify that this modification does not increase the agent's expected discounted cost of effort when a type-i task is available. In particular, this implies that the agent's expected discounted cost of effort decreases when any other type of task is available. Hence, this modification does not violate any of the constraints of $P^{(i)}$. The first-order effect of this modification on the value of the problem is

$$\begin{aligned}
&\epsilon\left(\pi_i'(e_i^{(i)})(1+\frac{\delta q_i}{1-\delta\lambda_i})^{-1}(1+\frac{\delta q_i}{1-\delta\lambda_i}) - \pi_j'(e_j^{(i)})\frac{1-\delta\lambda_i}{\delta q_j}\frac{\delta q_j}{1-\delta\lambda_i}\right)\\
&= \epsilon\left(\pi_i'(e_i^{(i)}) - \pi_j'(e_j^{(i)})\right) > 0,
\end{aligned}$$

where the inequality follows from the assumption that $e_j^{(i)} \geq e_i^{(i)}$ and the ordering of tasks. Thus, for a small enough ϵ this modification increases the value of the problem. □

Lemma 1 implies that the solution to $P^{(i)}$ is given by the solution to the following Lagrangian function:

$$\max_{(e_j)_{j\leq i},w} \pi_i(e_i) + \frac{1}{1-\delta\lambda_i}\left(\delta\sum_{j\leq i} q_j\pi_j(e_j) - w\right) - \mu\left\{e_i - \frac{1}{1-\delta\lambda_i}\left(g(w) - \delta\sum_{j\leq i} q_j e_j\right)\right\}.$$

Lemma 2. *For all $j \in \{1, ..., i\}$, $\pi_j'(e_j^{(i)}) \leq \frac{1}{g'(w^{(i)})}$ with equality if $e_j^{(i)} > 0$.*

This lemma, which follows directly from the FOCs of the concave Lagrangian, stipulates that the marginal cost of compensation is equal to the marginal productivity of effort from every implemented task. This intuitive property is less obvious than it seems. In particular, it relies on the fact that the auxiliary problem assumes the availability of its best admissible task in the initial period. On a technical level, this means that the most demanding incentive constraint appears, by construction, in the initial period. In particular, as we will show in the proof of Proposition 2, this implies that dispensing with the stationarity requirement in the definition of the auxiliary problems (requirement 3) does not benefit the principal. If the auxiliary problem did not begin with its best admissible task, then the optimal *stationary* solution would specify compensation and effort requirements for which the marginal cost of

compensation would be below the marginal productivity of effort, and more importantly, such a stationary solution would generally be suboptimal.

The next lemma ranks the agent's wage in the different auxiliary problems.

Lemma 3. *The sequence* $(w^{(1)}, w^{(2)}, \ldots, w^{(I)})$ *is strictly increasing.*

This lemma is the linchpin of the promotion-based dynamics that arises in the model. To develop some intuition, suppose by way of contradiction that $w^{(i+1)} \leq w^{(i)}$ and note that the combination of Lemma 2 and the concavity of $\pi_j(\cdot)$ and $g(\cdot)$ would then imply that $e_j^{(i+1)} \geq e_j^{(i)}$, for all $j \leq i$. Now, consider the continuation of the interaction in auxiliary problem $P^{(i+1)}$, which begins with the arrival of a task of type i. From the arrival of that task to the first arrival of a better task (i.e., a task of type l, for some $l \geq i+1$), the agent exerts weakly more effort on all tasks and receives a weakly lower wage in each period, compared to the solution to $P^{(i)}$. By Lemma 1, none of the constraints $IC_j^{(i+1)}$, $j \leq i$, is binding in the solution to $P^{(i+1)}$. Therefore, a periodic wage strictly lower than $w^{(i)}$ would suffice to incentivize (weakly) more effort than $\{e_j^{(i)}\}_{j=1}^{i}$ until the first arrival of a task that is better than task i. This contradicts the optimality of the solution to $P^{(i)}$.

Proof of Lemma 3. Suppose by way of contradiction that $w^{(i+1)} \leq w^{(i)}$. From the concavity of $g(\cdot)$ it follows that $g'(w^{(i+1)}) \geq g'(w^{(i)})$. Thus, for all $j \leq i$, Lemma 2 and the concavity of $\pi_j(\cdot)$ imply that $e_j^{(i)} \leq e_j^{(i+1)}$.

The binding constraint of $P^{(i)}$ can be written as

$$e_i^{(i)}(1 - \delta\lambda_i) = g(w^{(i)}) - \delta \sum_{k \leq i} q_k \cdot e_k^{(i)}.$$

Since $\lambda_{i+1} = \lambda_i + q_{i+1}$, we can similarly rewrite the binding constraint of $P^{(i+1)}$ as

$$e_{i+1}^{(i+1)}(1 - \delta\lambda_i) - \delta q_{i+1} \cdot e_{i+1}^{(i+1)} = g(w^{(i+1)}) - \delta \sum_{k \leq i} q_k \cdot e_k^{(i+1)} - \delta q_{i+1} \cdot e_{i+1}^{(i+1)}.$$

Adding $\delta q_{i+1} \cdot e_{i+1}^{(i+1)}$ to both sides and using the expression for the binding constraint of

$P^{(i)}$ and the consequences of the assumption at the beginning of the proof gives

$$e_{i+1}^{(i+1)}(1-\delta\lambda_i) = g(w^{(i+1)}) - \delta\sum_{k\leq i} q_k \cdot e_k^{(i+1)} \leq g(w^{(i)}) - \delta\sum_{k\leq i} q_k \cdot e_k^{(i)} = e_i^{(i)}(1-\delta\lambda_i),$$

which implies that $e_{i+1}^{(i+1)} \leq e_i^{(i)}$. Since $e_i^{(i)} \leq e_i^{(i+1)}$, it follows that $e_{i+1}^{(i+1)} \leq e_i^{(i+1)}$. However, as $\pi'_{i+1}(e) > \pi'_i(e)$ for all e, the only way this can occur without violating Lemma 2 is if $e_{i+1}^{(i+1)} = e_i^{(i+1)} = 0$. This, in turn, implies that $e_j^{(i+1)} = 0$ for all $j < i$, a solution that is not optimal due to our assumption that $\pi'_i(0) > \frac{1}{g'(0)}$. □

The combination of Lemmas 2 and 3, together with the concavity assumptions, delivers some simple but important comparison results *within* a given auxiliary problem and *across* different problems. Higher effort is exerted on better tasks within each auxiliary problem, and lower effort is exerted on tasks of a particular type in the higher-indexed problem (one that starts with a better task) than on tasks of that same type in a lower-indexed problem. These results are the basis for the main qualitative properties of the unique optimal contract, which we will define shortly.

Corollary 2. *Let $j \leq i$.*
1. For $j > 1$, $e_j^{(i)} \geq e_{j-1}^{(i)}$, with a strict inequality if $e_j^{(i)} > 0$, and
2. For $i < I$, $e_j^{(i)} \geq e_j^{(i+1)}$, with a strict inequality if $e_j^{(i)} > 0$.

3.2 The Optimal Contract

We now return to the original contracting problem and use the solutions of the auxiliary problems to define a particular contract, referred to as the *phase mechanism*. Later we prove that the phase mechanism is the essentially unique optimal contract.[5] Recall that $\mathscr{I}(h_t; i_t) = max\{i_s : s \leq t\}$ is the best type of task that has been available at least once along $(h_t; i_t)$.

Definition of the phase mechanism. *The job description and the compensation plan*

[5] We say that an optimal contract is essentially unique if all optimal contracts generate the same path of play for every realisation of project arrivals. That is, the optimal contract is unique up to a redundant multiplicity that exists off the path of play.

are, respectively,

$$work(h_t, i_t) = e_{i_t}^{(\mathscr{I}(h_t;i_t))}$$

and

$$pay(h_t, i_t, e_t) = \begin{cases} w^{(\mathscr{I}(h_t;i_t))} & \textit{if } e_s = work(h_s, i_s) \textit{ for all } s \leq t \\ 0 & \textit{otherwise.} \end{cases}$$

In words, the agent gets a positive wage only if he has followed his job description in the past. If he has done so, his wage and the required effort in every period are determined by the solution to the auxiliary problem indexed by the best task that has been available so far.

Under the phase mechanism, the relationship between the principal and the agent can be described using the metaphor of a ratchet that allows advancement only in the direction that favours the agent (Lemma 3 and Corollary 2). That is, for any realisation of task arrivals, the periodic wage and the effort exerted on every type of task are given by monotonic step functions. When the wage or effort requirements are updated, they jump to a new level where they stay until the next stochastic event causes another jump.

Proposition 2. *The phase mechanism is the essentially unique optimal contract.*

To develop some intuition for this result, consider the case of two possible types of tasks: a *low-productivity* task, $i = 1$, and a *high-productivity* task, $i = 2$. Suppose for a moment that the high-productivity task arrives in period 1. In this case, the solution to $P^{(2)}$ specifies an optimal incentive-compatible contract that is stationary over time.[6] Under this contract, the agent's expected payoff is zero and, as mentioned in Corollary 2, the effort exerted on a high-productivity task is *higher* than the effort exerted on a low-productivity task.

By construction, the agent's wage is identical in all periods. Therefore, while the agent's continuation utility is zero in periods in which a high-productivity task is available, his continuation utility is *strictly* positive whenever a low-productivity task is available. The positive continuation utility in these periods serves as an efficient compensation method for

[6] In the proof of Proposition 2 we show that due to the concavity assumptions, on the path of play, the "within-phase" solutions are necessarily stationary and, thus, the solution to $P^{(2)}$ corresponds to the optimal contract if the high-productivity task arrives first.

the high effort exerted on previous high-productivity tasks.

Now, suppose that the interaction does not begin with the high-productivity task. We refer to the time interval prior to the first arrival of the high-productivity task as *phase 1* of the interaction. The stationary solution to $P^{(2)}$ is still an incentive-compatible contract (for the whole interaction); however, during phase 1, the agent *strictly* prefers not to deviate from the contract. Therefore, in each period of this phase of the interaction, the principal can reduce the agent's wage and require more effort, without violating the incentive-compatibility constraints. In optimum, the wage is reduced and the effort required is increased until the incentive-compatibility constraints in phase 1 are binding, while the marginal productivity of effort is kept equal to the marginal cost of compensation (the resulting wage and effort correspond to the solution to $P^{(1)}$).

As a result, in phase 1, the marginal productivity of effort and the marginal cost of compensation are lower than they are after the first arrival of the high-productivity task (*phase 2*). This leads to the key observation that any additional modification "between" phases is either unprofitable or infeasible. For example, it is not in the interest of the principal to incentivize additional effort in phase 1 by increasing compensation in phase 2. On the other hand, even though the principal would benefit from increasing compensation in phase 1 in return for a higher effort on future high-productivity tasks (in phase 2), doing so is not feasible as it would violate the agent's incentive-compatibility constraint at the beginning of phase 2.

Proof of Proposition 2. First consider the auxiliary problems. For each $i \in \mathcal{I}$, define the *relaxed version of* $P^{(i)}$ to be identical to $P^{(i)}$ except that requirement (3) in the definition of the auxiliary setting of this problem is removed. We now show that the solution to $P^{(i)}$ and its relaxed version coincide. Recall that by Lemma 2 in the solution to $P^{(i)}$ the marginal cost of compensation (i.e., $\frac{1}{g'(\cdot)}$) equals the marginal productivity of effort for every task on which the agent exerts (positive) effort. Moreover, note that the relaxed version of $P^{(i)}$ is a convex maximisation problem in which the objective function is separable in all arguments. These observations jointly imply that if the solution to $P^{(i)}$ is not a solution to the relaxed version of $P^{(i)}$, then there exists an improvement of the following form: the required effort is modified at one particular history, and the wage offered immediately afterwards (in the

same period) is set at the lowest level under which all incentive-compatibility constraints are satisfied. Since $\pi'_j(e_j^{(i)}) \leq \frac{1}{g'(w^{(i)})}$ (Lemma 2), and $\pi'(\cdot)$ and $g'(\cdot)$ are decreasing, every such modification where the agent's effort is increased strictly reduces the total expected value for the principal. Similarly, since the above inequality holds with equality for $j \leq i$ such that $e_j^{(i)} > 0$, every such modification where the agent's effort is decreased also strictly reduces the total expected value for the principal. Therefore, the solution to $P^{(i)}$ is a solution to the relaxed version of $P^{(i)}$, and it is unique since the problem is convex.

Now consider the general optimisation problem. Denote by C_0 the class of all incentive-compatible contracts for which, whenever a task that is better than all previously available tasks arrives, the agent's continuation utility (before exerting effort) is zero. The phase mechanism is the essentially unique optimal contract in the set C_0. To see this, notice that the restriction to contracts in C_0 implies that it is sufficient to show that the phase mechanism attains the highest expected value between any two (subsequent) earliest arrivals of tasks that are superior to all previously available ones. But this follows directly from the construction of the phase mechanism and the argument in the first step of the proof.

Finally, suppose that the phase mechanism is suboptimal in the class of *all* incentive-compatible contracts. Since the principal solves a convex optimisation problem that is separable in all arguments, it follows from the claim established in the previous paragraph that there exists a profitable modification where (1) for some $(h_t; i_t)$ in phase $k < I$, the phase mechanism is marginally altered in period t in the direction that reduces the agent's payoff in that period (i.e., either the required effort is increased or the wage is decreased), and (2) at $(h_{t'}; i_{t'})$, which is part of phase $k' > k$ and where $h_{t'}$ is a continuation of $(h_t; i_t)$, the phase mechanism is marginally altered to restore incentive compatibility. Since under the phase mechanism the marginal cost of compensation and the marginal benefit from effort during phase k are below those of phase $k' > k$, any such modification reduces the principal's expected payoff, a contradiction.

The optimal contract is essentially unique due to the concavity of the objective function and the convexity of the constraints. □

3.3 The Commitment Assumption: Discussion

Proposition 2 relies on the assumption that the principal can commit to any contract while the agent reoptimises his continuation play after any history. In this section, we discuss the

role of the commitment assumption by briefly considering two alternative specifications.

To assess whether the need to provide ongoing incentives to the agent is consequential in terms of efficiency, we consider a benchmark where the agent decides whether or not to accept the contract at time zero—a decision to which he commits. We refer to the principal's optimal contract under the benchmark where both players have commitment power as the *first-best contract*. It is straightforward to verify that this contract is stationary and is fully described by a vector of required efforts $(e_1^{fb}, ..., e_I^{fb})$ and a wage w^{fb} that together maximise the (static) profit $\sum_{i=1}^{I} q_i \pi_i(e^{fb}) - w^{fb}$ under the constraint $g(w^{fb}) = \sum_{i=1}^{I} q_i e_i^{fb}$. Note that since $\pi_i'(e) < \pi_{i+1}'(e)$ for all $e \geq 0$ and $i < I$, it follows that if $e_{i+1}^{fb} > 0$, then $e_i^{fb} < e_{i+1}^{fb}$.

The key distinction between the first-best contract and the phase mechanism is in the relative timing of compensation and effort. The only constraint faced by the principal when both players can commit is to provide sufficient compensation in expectation, whereas, if the agent can walk away from the interaction at any moment, compensation cannot be provided for effort that has not yet been exerted. More formally, while positive as well as negative expected continuation utility are perfectly legitimate under the first-best contract, the phase mechanism is optimal under the additional constraint by which the agent's continuation utility must always be nonnegative. We now show how, relative to the first-best contract, this additional constraint leads to *lower* wages and *over-provision* of effort in lower-indexed phases and *higher* wages and *under-provision* of effort in higher-indexed phases.

Since $e_I^{fb} > \sum_{i=1}^{I} q_i e_i^{fb}$, the agent has an incentive to renege on the first-best contract if task I is available. Thus, for the phase mechanism to be incentive compatible when task I arrives, it must be the case that $w^{(I)} > w^{ef}$ and $e_i^{(I)} < e_i^{ef}$ for all $i \in \mathcal{I}$. Similarly, as $e_1^{fb} < \sum_{i=1}^{I} q_i e_i^{fb}$, the agent would strictly benefit from this contract if only tasks of type 1 were to arrive. Hence, in phase 1 of the phase mechanism the agent receives less wage and is required to exert more effort on tasks of type 1 relative to the first-best contract. More generally, by Corollary 2, under the phase mechanism, the agent's effort on every type of task is decreasing in the phase of the contract. Therefore, relative to the first-best contract, so long as the index of the best-to-date task is sufficiently low, the phase mechanism leads to

over-provision of effort in return for a lower wage, but, eventually, there is under-provision of effort and the wage is higher.

To conclude this section, we discuss when the phase mechanism is an equilibrium when neither the agent nor the principal can commit. To do so, we consider the principal's preferred equilibrium in the dynamic stochastic game that corresponds to our model. In general, characterising the equilibria of a dynamic stochastic game is a difficult problem. However, due to the promotion-based dynamics of the phase mechanism, it is straightforward to check whether the phase mechanism is an equilibrium.

When the phase mechanism changes phases, the agent's wage increases and his required effort on every type of task decreases. Hence, if the principal has an incentive to renege on the phase mechanism (at some history) he has an incentive to do so in phase[7] I. By deviating in a period within phase I, the principal saves the cost of providing wage in that period, $w^{(I)}$, but forgoes his expected discounted profit from future periods, $\sum_{t=1}^{\infty} \delta^t \left(\sum_{i \in \mathcal{I}} q_i \pi_i(e_i^{(I)}) - w^{(I)} \right)$. Thus, the phase mechanism is an equilibrium if

$$w^{(I)} \leq \delta \sum_{i \in \mathcal{I}} q_i \pi_i(e_i^{(I)}),$$

in which case, since the phase mechanism is the optimal contract when the principal has commitment power, it is clearly the principal's preferred equilibrium.

4 Conclusion

We study a stylised dynamic contracting problem where the worker's tasks vary over time. Our main result shows that even when task arrival is i.i.d. across periods, the worker's career path exhibits seniority-based dynamics. As times goes by, the worker's expected wage increases while his expected workload decreases as a result of a rank-based contract.

Within each rank, the worker's wage is constant; however, his actual effort varies according to the random arrival of tasks. Upon the arrival of a task that is better than all

[7] Without loss of generality we assume that following a deviation (by either player) the agent does not exert effort and the principal does not provide wage.

previously available tasks, the worker is "promoted" to a new rank, he exerts less effort on tasks of lower quality (those that have been implemented in the past), and his periodic wage is higher. Under this contract, promotions are not a consequence of new outside offers, additional information about the quality/productivity of the worker, a contest of any form, or an efficient back-loading of compensation to overcome informational frictions. Instead, promotions arise entirely as an efficient response to the variation in a worker's tasks.

The rank structure of the optimal contract also implies that random shocks to the worker's productivity have a limited impact on his wage and job requirements when the employer can offer long-term contracts. The simplest manifestation of this stickiness is that the arrival of any task that is worse than the task that was available in the period in which the worker was hired has no impact on his wage or job requirements. Thus, even the wage of a recently hired worker can exhibit considerable stickiness if he was hired in a period in which productivity was high. More generally, the worker's rank depends only on the best-to-date task and so, as time goes by, the probability that the realised task will change his rank decreases. Hence, as the worker's tenure at a firm increases so does the stickiness of his wage and work requirements.

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